Mathematical Induction (selected questions)

1. (a) Let P(n) be the proposition : $1+2+\cdots+n=\frac{1}{2}n(n+1)$

For P(1), L.H.S.= $1=\frac{1}{2}\times 1\times(1+1)$=R.H.S., ∴P(1) is true.

Assume P(k) is true for some natural number k, that is, $1+2+\cdots+k=\frac{1}{2}k(k+1)$………(1)

For P(k + 1), $1+2+\cdots+k+(k+1)=\frac{1}{2}k(k+1)+(k+1)$, by (1)

$=\frac{1}{2}(k+1)[k+2]$

$=\frac{1}{2}(k+1)[(k+1)+1]$

∴ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true for all natural numbers n.

1. (b) Let P(n) be the proposition : $1^2+2^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$

For P(1), L.H.S.= $1^2=1=\frac{1}{6}\times 1\times(2+1)(2\times 1+1)$=R.H.S., ∴ P(1) is true.

Assume P(k) is true for some k∈ **N**, that is, $1^2+2^2+\cdots+k^2=\frac{1}{6}k(k+1)(2k+1)$………(1)

For P(k + 1), $1^2+2^2+\cdots+k^2+(k+1)^2=\frac{1}{6}k(k+1)(2k+1)+(k+1)^2$, by (1)

$=\frac{1}{6}(k+1)[k(2k+1)+6(k+1)]$

$=\frac{1}{6}(k+1)[2k^2+7k+6]$

$=\frac{1}{6}(k+1)(k+2)(2k+3)$

$=\frac{1}{6}(k+1)[(k+1)+1][2(k+1)+1]$

∴ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true ∀ n ∈ **N.**

1. (c) Let P(n) be the proposition :"$x^{2n}-y^{2n}$ is divisible by x + y for any integers x, y and positive integer n."

or "$x^{2n}-y^{2n}=(x+y)g_n(x,y)$, where $g_n(x,y)$ is a polynomial in x ,y and x, y ∈ **Z**, n∈**N**.

For P(1), $x^2-y^2=(x+y)(x-y)=(x+y)g_1(x,y)$, ∴ P(1) is true.

Assume P(k) is true for some k∈ **N**, that is,

$x^{2k}-y^{2k}=(x+y)g_k(x,y)$, where $g_k(x,y)$ ∈ **Z**[x, y]

For P(k + 1), $x^{2(k+1)}-y^{2(k+1)}=[x^{k+1}-y^{k+1}][x^{k+1}+y^{k+1}]$

$=(x+y)g_k(x,y)[x^{k+1}+y^{k+1}]$

$=(x+y)g_{k+1}(x,y)$, where $g_{k+1}(x,y)=g_k(x,y)[x^{k+1}+y^{k+1}]$∈ **Z**[x, y]

∴ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true ∀ n ∈ **N.**

1. (d) Let P(n) be the proposition : $1^3+2^3+\cdots+n^3=\left[\frac{1}{2}n(n+1)\right]^2$

For P(1), L.H.S.= $1^3=1=\left[\frac{1}{2}\times 1\times(1+1)\right]^2$=R.H.S. , ∴ P(1) is true.

Assume P(k) is true for some k∈ **N**, that is, $1^3+2^3+\cdots+k^3=\left[\frac{1}{2}k(k+1)\right]^2$………(1)

For P(k + 1), $1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \left[\frac{1}{2}k(k+1)\right]^2 + (k+1)^3$, by (1)

$= \frac{1}{4}(k+1)^2[k^2 + 4(k+1)]$
$= \frac{1}{4}(k+1)^2[\,k^2 + 4\,k + 4]$
$= \frac{1}{4}(k+1)^2[\,k+1]^2$

$= \left\{\frac{1}{2}(k+1)[(k+1)+1]\right\}^2$

∴ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true ∀ n ∈ **N.**

1. **(e)** Let P(n) be the proposition : $2 \times 1 + 3 \times 2 + 4 \times 2^2 + 5 \times 2^3 + \cdots + (n+1)2^{n-1} = n2^n$

For P(1), $2 \times 1 = 2 = 1 \times 2^1$, ∴ P(1) is true.

Assume P(k) is true for some k∈ **N** , that is, $2 \times 1 + 3 \times 2 + 4 \times 2^2 + 5 \times 2^3 + \cdots + (k+1)2^{k-1} = k2^k$ ... (1)

For P(k + 1), $2 \times 1 + 3 \times 2 + 4 \times 2^2 + 5 \times 2^3 + \cdots + (k+1)2^{k-1} + (k+2)2^k$

$= k2^k + (k+2)2^k$, by (1)

$= [k + (k+2)]2^k = 2(k+1)2^k = (k+1)2^{k+1}$. ∴ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true ∀ n ∈ **N.**

1. **(f)** Let P(n) be the proposition : $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}$

For P(1), $1 - \frac{1}{2} = \frac{1}{2} = \frac{1}{1+1}$, ∴ P(1) is true.

Assume P(k) is true for some k∈ **N** , that is, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2k} = \frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k}$ ... (1)

For P(k + 1), $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2k} + \frac{1}{2k+1} - \frac{1}{2(k+1)} = \left[\frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k}\right] + \frac{1}{2k+1} - \frac{1}{2(k+1)}$, by (1)

$= \frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k} + \left[\frac{1}{k+1} + \frac{1}{2k+1} - \frac{1}{2(k+1)}\right] = \frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2(k+1)}$

. ∴ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true ∀ n ∈ **N.**

1. **(g)** Let P(n) be the proposition : $2^{4n} - 1 = 15a_n$ where $a_n \epsilon$**N.**

For P(1), $2^4 - 1 = 15 = 15a_1$, where $a_1 = 1\epsilon$**N.** ∴ P(1) is true.

Assume P(k) is true for some k∈ **N** , that is, $2^{4k} - 1 = 15a_k$ ... (1) , where $a_k \epsilon$**N.**

For P(k + 1), $2^{4(k+1)} - 1 = 16 \times 2^{4k} - 1 = 16 \times (2^{4k} - 1) + 15 = 16 \times (15a_k) + 15$, by (1)

$= 15(16a_k + 1) = 15a_{k+1}$, where $a_{k+1} \epsilon$**N.**

∴ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true ∀ n ∈ **N.**

1. **(h)** Let P(n) be the proposition : The no. of diagonals in a convex n-sided polygon is $f(n) = \frac{1}{2}n(n-3)$ $(n \geq 3)$.

For P(3), There is no diagonal in a triangle. f(3) = 0. ∴ P(3) is true.

Assume P(k) is true for some $k \in \mathbf{N} \setminus \{1,2\}$, that is,

The no. of diagonals in a convex k-sided polygon is $f(k)=\frac{1}{2}k(k-3)$ $(k \geq 3)$ ………(1)

For P(k + 1), Let $P_1, P_2, \ldots, P_k, P_{k+1}$ be the vertices in clockwise direction of the (k+1)-sided polygon.

By (1), The no. of diagonals in a convex k-sided polygon $P_1P_2\ldots P_k$ is $f(k)=\frac{1}{2}k(k-3)$

There are additional $(k-1)$ diagonals connected to the the point P_{k+1}, namely, $P_{k+1}P_2, P_{k+1}P_3, \ldots, P_{k+1}P_k$.

$\therefore$ The no. of diagonals in a convex (k+1)-sided polygon $P_1P_2\ldots P_kP_{k+1}$ is

$$f(k) + (k-1) = \frac{1}{2}k(k-3)+(k-1) = \frac{1}{2}\{k(k-3)+2(k-1)\} = \frac{1}{2}\{k^2-k-2\} = \frac{1}{2}\{(k+1)[(k+1)-3]\}$$

$= f(k + 1)$. $\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall\, n \in \mathbf{N}$.

1. (i) Let P(n) be the proposition: $m, n \geq 2$, $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_m \in \mathbf{R}$,

$$(a_1+a_2+\cdots+a_n)(b_1+b_2+\cdots+b_m) = \sum_{j=1}^m \sum_{i=1}^n a_ib_j = \sum_{i=1}^n \sum_{j=1}^m a_ib_j$$

For P(2), $(a_1+a_2)(b_1+b_2+\cdots+b_m) = (a_1b_1+a_2b_1)+(a_1b_2+a_2b_2)+\cdots+(a_1b_m+a_2b_m)$

$= \sum_{i=1}^2 a_ib_1 + \sum_{i=1}^2 a_ib_2 + \cdots + \sum_{i=1}^n a_ib_m = \sum_{j=1}^m \sum_{i=1}^2 a_ib_j$

Also, $(a_1+a_2)(b_1+b_2+\cdots+b_m) = (a_1b_1+a_1b_2+\cdots+a_1b_m)+(a_2b_1+a_2b_2+\cdots+a_2b_m)$

$= \sum_{j=1}^m a_1b_j + \sum_{j=1}^m a_2b_j = \sum_{i=1}^2 \sum_{j=1}^m a_ib_j$

$\therefore$ P(2) is true.

Assume P(k) is true for some $k \in \mathbf{N}\setminus\{1\}$,

$(a_1+a_2+\cdots+a_k)(b_1+b_2+\cdots+b_m) = \sum_{j=1}^m \sum_{i=1}^k a_ib_j = \sum_{i=1}^k \sum_{j=1}^m a_ib_j$ ... (1)

For P(k + 1), $(a_1+a_2+\cdots+a_k+a_{k+1})(b_1+b_2+\cdots+b_m)$

$= (a_1'+a_2'+\cdots+a_k')(b_1+b_2+\cdots+b_m)$ where $a_1' = a_1, \ldots, a_{k-1}' = a_{k-1}, a_k' = a_k + a_{k+1}$

$= \sum_{j=1}^m \sum_{i=1}^k a_i'b_j = \sum_{j=1}^m [a_1b_j + a_2b_j + \cdots + a_{k-1}b_j + (a_k+a_{k+1})b_j]$, by (1)

$= \sum_{j=1}^m [a_1b_j + a_2b_j + \cdots + a_kb_j + a_{k+1}b_j] = \sum_{j=1}^m \sum_{i=1}^{k+1} a_ib_j$

Also, $(a_1+a_2+\cdots+a_k+a_{k+1})(b_1+b_2+\cdots+b_m)$

$= (a_1'+a_2'+\cdots+a_k')(b_1+b_2+\cdots+b_m)$ where $a_1' = a_1, \ldots, a_{k-1}' = a_{k-1}, a_k' = a_k + a_{k+1}$

$= \sum_{i=1}^k \sum_{j=1}^m a_i'b_j = \sum_{j=1}^m a_1b_j + \sum_{j=1}^m a_2b_j + \cdots + \sum_{j=1}^m a_{k-1}b_j + \sum_{j=1}^m (a_k+a_{k+1})b_j$

$= \sum_{j=1}^m a_1b_j + \sum_{j=1}^m a_2b_j + \cdots + \sum_{j=1}^m a_{k-1}b_j + \sum_{j=1}^m (a_kb_j + a_{k+1}b_j)$

$= \sum_{j=1}^m a_1b_j + \sum_{j=1}^m a_2b_j + \cdots + \sum_{j=1}^m a_{k-1}b_j + + \sum_{j=1}^m a_kb_j + \sum_{j=1}^m a_{k+1}b_j$

$= \sum_{i=1}^{k+1} \sum_{j=1}^m a_ib_j$

$\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall\, n \in \mathbf{N}\setminus\{1\}$.

1. (j) Let P(n) be the proposition : $3^{n-2} \geq n^5$ for $n \geq 20$.

For P(20), $3^{20-2} = 3^{18} = (3^3)^6 = 27^6 \geq 20^6 \geq 20^5$. $\therefore$ P(20) is true.

Assume P(k) is true for some $k \in \mathbf{N}$, where $k \geq 20$, that is, $3^{k-2} \geq k^5$ ………(1)

For P(k + 1), $3^{(k+1)-2} = 3(3^{k-2}) \geq 3(k^5)$, by (1)

$= k^5 + 3k(k^4) \geq k^5 + 0.4\,k\,k^4 + 0.4\,k^2k^3 + 0.4\,k^3\,k^2 + 0.4\,k^4\,k + 0.4\,k^5$

$\geq k^5 + 0.4(20)k^4 + 0.4(20)^2k^3 + 0.4(20)^3k^2 + 0.4(20)^4k + 0.4\,(20)^5$, since $k \geq 20$.

$\geq k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1$

$= (k + 1)^5$ $\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall\, n \in \mathbf{N}$, where $n \geq 20$.

1. **(k)** Let P(n) be the proposition : $2^n > n^2$ for $n \geq 5$.

For P(5), $2^5 = 32 > 25 = 5^2$ $\therefore$ P(5) is true.

Assume P(k) is true for some $k \in \mathbf{N}$, where $k \geq 5$, that is $2^k > k^2$ ... (1)

For P(k + 1), $2^{k+1} = 2(2^k) > 2k^2 = k^2 + k^2 \geq k^2 + 5k = k^2 + 2k + 3k$

$\geq k^2 + 2k + 3(5) > k^2 + 2k + 1 = (k+1)^2$ $\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall\, n \in \mathbf{N}$, where $n \geq 5$.

1. **(l)** Let P(m) be the proposition : $\sum_{n=1}^{m} \frac{n(n+1)(n+2)\ldots(n+p-1)}{p!} = \frac{m(m+1)(m+2)\ldots(m+p)}{(p+1)!}$

For P(1), $\frac{(1)(2)(3)\ldots(p)}{p!} = 1 = \frac{(1)(2)(3)\ldots(p)(1+p)}{(p+1)!}$ $\therefore$ P(1) is true.

Assume P(k) is true for some $k \in \mathbf{N}$, that is $\sum_{n=1}^{k} \frac{n(n+1)(n+2)\ldots(n+p-1)}{p!} = \frac{k(k+1)(k+2)\ldots(k+p)}{(p+1)!}$ ... (1)

For P(k + 1), $\sum_{n=1}^{k+1} \frac{n(n+1)(n+2)\ldots(n+p-1)}{p!} = \sum_{n=1}^{k} \frac{n(n+1)(n+2)\ldots(n+p-1)}{p!} + \frac{(k+1)(k+2)\ldots(k+1+p-1)}{p!}$

$= \frac{k(k+1)(k+2)\ldots(k+p)}{(p+1)!} + \frac{(k+1)(k+2)\ldots(k+p)}{p!}$, by (1)

$= \frac{(k+1)(k+2)\ldots(k+p)}{(p+1)!}[k + (p+1)] = \frac{(k+1)(k+2)\ldots(k+p)[(k+1)+p]}{(p+1)!}$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(m) is true $\forall\, m \in \mathbf{N}$.

1. **(m)** Let P(n) be the proposition : $C_r^n = \frac{n!}{r!(n-r)!}$ is always an integer, where $0 \leq r \leq n$

For P(0), $C_0^0 = 1$, which is an integer.

Assume P(k) is true for some $k \in \mathbf{N} \cup \{0\}$, that is, $C_r^k = \frac{k!}{r!(n-r)!}$ is always an integer. …(1)

For P(k + 1), $C_0^{k+1} = C_{k+1}^{k+1} = 1$ are integers.

For $r \geq 1$, $C_r^{k+1} = C_r^k + C_{r-1}^k$ is the sum of two integers by (1), and is an integer.

$\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall\, n \in \mathbf{N} \cup \{0\}$.

1. **(n)** Let P(n) be the proposition : $2 \times 4^{2n+1} + 3^{3n+1} = 11a_n$, where $a_n \in \mathbf{N}$

For P(1), $2 \times 4^3 + 3^4 = 209 = 11 \times 19 = 11a_1$ $\therefore$ P(1) is true.

Assume P(k) is true for some k$\in$ **N** , that is, $2 \times 4^{2k+1} + 3^{3k+1} = 11a_k \ldots (1)$, where $a_k \in$ **N**

For P(k+1), $2 \times 4^{2k+3} + 3^{3k+4} = 32\times 4^{2k+1} + 27 \times 3^{3k+1}$

$= 16(2 \times 4^{2k+1} + 3^{3k+1}) + 11 \times 3^{3k+1}$

$= 16(11a_k) + 11 \times 3^{3k+1}$, by (1)

$= 11(16a_k + 3^{3k+1}) = 11a_{k+1}$, where $a_{k+1} = 16a_k + 3^{3k+1} \in$ **N.**

$\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall$ n $\in$ **N.**

1. (o) Let P(n) be the proposition : $4^{2n+1} + 3^{n+2} = 13a_n$, where $a_n \in$ **N** .

For P(1), $4^3 + 3^3 = 91 = 13\times7 = 13a_1$ $\therefore$ P(1) is true.

Assume P(k) is true for some k$\in$ **N** , that is, $4^{2k+1} + 3^{k+2} = 13a_k \ldots (1)$, where $a_k \in$ **N**

For P(k+1), $4^{2k+3} + 3^{k+3} = 16\times 4^{2k+1} + 3 \times 3^{k+2}$

$= 3(4^{2k+1} + 3^{k+2}) + 13 \times 4^{2k+1}$

$= 3(13a_k) + 13 \times 4^{2k+1}$, by (1)

$= 13(3a_k + 3^{3k+1}) = 13a_{k+1}$, where $a_{k+1} = 3a_k + 3^{3k+1} \in$ **N**

$\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall$ n $\in$ **N.**

1. (p) Let P(n) be the proposition : $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < 1 - \frac{1}{2n+1}$ (We change the proposition and prove this first.)

For P(1), $\frac{1}{1+1} = \frac{1}{2} < \frac{2}{3} = 1 - \frac{1}{2(1)+1}$ $\therefore$ P(1) is true.

Assume P(k) is true for some k$\in$ **N** , that is, $\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k} < 1 - \frac{1}{2k+1}$ ... (1)

For P(k+1), $\frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2(k+1)} = \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k}\right) + \left(\frac{1}{2k+1} + \frac{1}{2(k+1)} - \frac{1}{k+1}\right)$, by (1)

$$< 1 - \frac{1}{2k+1} + \left(\frac{1}{2k+1} - \frac{1}{2(k+1)}\right) = 1 - \frac{1}{2(k+1)} = 1 - \frac{1}{2k+2} < 1 - \frac{1}{2k+3}$$

$\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall$ n $\in$ **N.**

$\therefore \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < 1 - \frac{1}{2n+1} < 1$

1. (q) Let P(n) be the proposition : The number of pairs of non-negative integers (x, y) satisfying $x + 2y = n$

is $f(n) = \frac{1}{2}(n+1) + \frac{1}{4}[1 + (-1)^n]$

For P(1), $(x, y) = (1, 0)$, $f(1) = 1$ $\therefore$ P(1) is true.

For P(2), $(x, y) = (2, 0)$ and $(0, 1)$ are solutions. Also, $f(2) = 2$ $\therefore$ P(2) is true.

Assume P(k) is true for some k$\in$ **N**, that is,

The number of pairs of non-negative integers (x, y) satisfying $x + 2y = k$ is

$$f(k) = \frac{1}{2}(k+1) + \frac{1}{4}\left[1 + (-1)^k\right] \qquad \text{.........(1)}$$

For P(k + 2), Consider the equation $x + 2z = k$ ....(2)

The number of pairs of non-negative integers (x, z) satisfying (2) is $f(k) = \frac{1}{2}(k+1) + \frac{1}{4}\left[1 + (-1)^k\right]$, by (1)

Put $z = y - 1$, (2) becomes $x + 2(y - 1) = k$ or $x + 2y = k + 2$ ....(3)

Obviously $(x, y) = (k + 2, 0)$ satisfies (3), but $(x, z) = (x, y - 1) = (k + 2, -1)$ does not satisfy (2).

$\therefore$ The total number of pairs of non-negative integers (x, y) satisfying (3) is

$$1 + f(k) = 1 + \frac{1}{2}(k+1) + \frac{1}{4}\left[1 + (-1)^k\right] = \frac{1}{2}\left[(k+2)+1\right] + \frac{1}{4}\left[1 + (-1)^{k+2}\right] = f(k+2)$$

$\therefore$ P(k + 2) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall\ n \in \mathbf{N}$.

2. **(a)** Let P(n) be the proposition : $\left(\sqrt{3}+1\right)^{2n+1} - \left(\sqrt{3}-1\right)^{2n+1} = 2^{n+1}a_n$, where $a_n \in \mathbf{Z}$.

For P(0), $\left(\sqrt{3}+1\right)^1 - \left(\sqrt{3}-1\right)^1 = 2 = 2a_0$, where $a_0 = 1$

For P(1), $\left(\sqrt{3}+1\right)^3 - \left(\sqrt{3}-1\right)^3 = 2\left[3\left(\sqrt{3}\right)^2 + 1\right] = 20 = 2^{1+1}a_1$, where $a_1 = 5$.

Assume P(k – 1) and P(k) are true for some $k \in \mathbf{N}$ $(k \geq 1)$, that is,

$$\left(\sqrt{3}+1\right)^{2k-1} - \left(\sqrt{3}-1\right)^{2k-1} = 2^k a_{k-1}, \qquad \left(\sqrt{3}+1\right)^{2k+1} - \left(\sqrt{3}-1\right)^{2k+1} = 2^{k+1}a_k \quad \text{....(*)}$$

For P(k + 1), $\left(\sqrt{3}+1\right)^{2k+3} - \left(\sqrt{3}-1\right)^{2k+3}$

$$= \left[\left(\sqrt{3}+1\right)^{2k+1} - \left(\sqrt{3}-1\right)^{2k+1}\right]\left[\left(\sqrt{3}+1\right)^2 + \left(\sqrt{3}-1\right)^2\right] - \left(\sqrt{3}-1\right)^2\left(\sqrt{3}+1\right)^{2k+1} + \left(\sqrt{3}+1\right)^2\left(\sqrt{3}-1\right)^{2k+1}, \text{ by (*)}$$

$$= 2^{k+1}a_k \times 8 - \left(\sqrt{3}+1\right)^2\left(\sqrt{3}-1\right)^2\left[\left(\sqrt{3}+1\right)^{2k-1} - \left(\sqrt{3}-1\right)^{2k-1}\right]$$

$$= 2^{k+2}a_k \times 4 - 8 \times 2^k a_{k-1} = 2^{k+2}(4a_k - a_{k-1}) = 2^{k+2}a_{k+1}, \quad \text{where } a_{k+1} = 4a_k - a_{k-1} \in \mathbf{Z}.$$

$\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall\ n \in \mathbf{N} \cup \{0\}$.

2. **(b)** Let $\alpha = 3 + \sqrt{5}$, $\beta = 3 - \sqrt{5}$, then $\alpha + \beta = 6$, $\alpha\beta = 4$.

Let P(n) be the proposition : $\alpha^n + \beta^n = 2^n a_n$, where $a_n \in \mathbf{Z}$.

For P(1), $\alpha + \beta = 6 = 2^1 a_1$, where $a_1 = 3 \in \mathbf{N}$.

For P(2), $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = 6^2 - 2 \times 4 = 28 = 2^2 a_2$, where $a_2 = 7 \in \mathbf{Z}$.

Assume P(k) and P(k + 1) are true, that is,

$$\alpha^k + \beta^k = 2^k a_k, \ \alpha^{k+1} + \beta^{k+1} = 2^{k+1}a_{k+1} \ \dots (*) \ \text{, where } a_k, a_{k+1} \in \mathbf{Z}.$$

For P(k + 2), $\alpha^{k+2} + \beta^{k+2} = \left(\alpha^{k+1} + \beta^{k+1}\right)(\alpha + \beta) - (\alpha\beta)\left(\alpha^k + \beta^k\right) = \left(2^{k+1}a_{k+1}\right)(6) - (4)\left(2^k a_k\right)$

$= 2^{k+2}(3a_{k+1} - a_k) = 2^{k+2}a_{k+2}$, where $a_{k+2} \in \mathbf{Z}$.

$\therefore$ P(k+2) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall\ n \in \mathbf{N}$.

3. Let P(n) be the proposition : There are two collections of marbles of the same quantity, n. According to the rules of taking marbles set by the question, the first player wins.

For P(1), The first player take 1 marble from one collection and the second player takes the last marble and wins.

$\therefore$ P(1) is true.

Assume P(1), P(2), …, P(k) are true. So if there are two collections of marbles of the same quantity less than or equal to k, the second player wins.

For P(k + 1), There are two collections of marbles of the same quantity, k + 1.

If the first player takes away any number of marbles, say $p, (1 \le p \le k+1)$ from one collection. The second player takes away also p marbles from the other collection. So now we have two collections of marbles each of $k + 1 - a \le k$. Thus by the inductive hypothesis, the second player wins. $\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall\ n \in \mathbf{N}$.

4. **(a)** Let P(n) be the proposition : $\left(\sqrt{m^2+1} - m\right)^n = a_n\sqrt{m^2+1} - b_n$, where $a_n, b_n \in \mathbf{Z}$, $m \in$ N.

For P(1), $\sqrt{m^2+1} - m = a_1\sqrt{m^2+1} - b_1$, where $a_1 = 1$, $b_1 = m$.

For P(2), $\left(\sqrt{m^2+1} - m\right)^2 = 2m^2 + 1 - 2m\sqrt{m^2+1} = a_2\sqrt{m^2+1} - b_2$, where $a_2 = -2m$, $b_2 = -(2m^2 + 1)$.

$\therefore$ P(1) , P(2) are true.

Assume P(k) is true for some $k \in \mathbf{N}$, that is, $\left(\sqrt{m^2+1} - m\right)^k = a_k\sqrt{m^2+1} - b_k$ ………(1)

For P(k + 1), $\left(\sqrt{m^2+1} - m\right)^{k+1} = \left(\sqrt{m^2+1} - m\right)^k\left(\sqrt{m^2+1} - m\right)$

$$= \left(a_k\sqrt{m^2+1} - b_k\right)\left(\sqrt{m^2+1} - m\right) = (-a_k m - b_k)\sqrt{m^2+1} - \left[-\left(a_k(m^2+1) + b_k m\right)\right] = a_{k+1}\sqrt{m^2+1} - b_{k+1}$$

where $a_{k+1} = -(a_k m + b_k)$, $b_{k+1} = -\left(a_k m^2 + a_k + b_k m\right)$ ……….(2)

$\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall\ n \in \mathbf{N}$.

(b) From (2), $a_n = -(a_{n-1}m + b_{n-1})$, $b_n = -\left(a_{n-1}m^2 + a_{n-1} + b_{n-1}m\right)$ ……….(3)

From (3), $a_n^2(m^2+1) - b_n^2 = (-a_{n-1}m - b_{n-1})^2(m^2+1) - \left(a_{n-1}m^2 + a_{n-1} + b_{n-1}m\right)^2$

$$= b_{n-1}^2 - a_{n-1}(m^2+1) = -\left[a_{n-1}(m^2+1) - b_{n-1}^2\right] = (-1)^2\left[a_{n-2}(m^2+1) - b_{n-2}^2\right] = \ldots$$

$$= (-1)^{n-1}\left[a_1(m^2+1) - b_1^2\right] = (-1)^{n-1}\left[1(m^2+1) - m^2\right] = (-1)^{n-1}$$ ……….(4)

From (3),

(i) If n is even, $a_n < 0$, $b_n < 0$. $a_n{}^2(m^2+1) - b_n{}^2 = (-1)^n = -1$

$\therefore$ $a_n{}^2(m^2+1) + 1 = b_n{}^2$ ……….(5)

$\left(\sqrt{m^2+1} - m\right)^n = a_n\sqrt{m^2+1} - b_n = -b_n - a_n\sqrt{m^2+1} = \sqrt{b_n{}^2} - \sqrt{a_n{}^2(m^2+1)} = \sqrt{N+1} - \sqrt{N}$, by (5).

(ii) If n is odd, $a_n > 0$, $b_n > 0$. $a_n{}^2(m^2+1) - b_n{}^2 = (-1)^n = 1$

$\therefore$ $a_n{}^2(m^2+1) = b_n{}^2 + 1$ ……….(6)

$\left(\sqrt{m^2+1} - m\right)^n = a_n\sqrt{m^2+1} - b_n = \sqrt{a_n{}^2(m^2+1)} - \sqrt{b_n{}^2} = \sqrt{N+1} - \sqrt{N}$, $N \in \mathbf{N}$, by (6)

6. First, show by Math. Induction the proposition $P(n) : 1 + 3 + \ldots + (2n-1) = n^2 \quad \forall n \in \mathbf{N}$ ….(1)

Then,

$a_1 = 1 \qquad\qquad = 1 = 1^3$

$a_2 = 3 + 5 \qquad\qquad = 8 = 2^3$

$a_3 = 7 + 9 + 11 \qquad\qquad = 27 = 3^3$

: : : : :

$a_n = (n^2 - n + 1) + (n^2 - n + 3) + \ldots + [n^2 - n + (2n-1)] \qquad = n^3$

The last term of $a_n = n^2 + n - 1 = 2\left(\frac{n(n+1)}{2}\right) - 1$.

Adding up all equalities, $\sum_{r=1}^{n} r^3 = 1 + 3 + 5 + \ldots + \left[2\left(\frac{n(n+1)}{2}\right) - 1\right] = \left(\frac{n(n+1)}{2}\right)^2 = \frac{1}{4}n^2(n+1)^2$,by (1).

$b_1 = 1$ (no. of term = 1^2), $b_2 = 3 + 5 + 7 + 9$ (no. of terms = 2^2), $b_3 = 11 + 13 + 15 + \ldots + 27$ (no. of terms = 3^2),

The last term of $b_n = 2[1^2 + 2^2 + \ldots + n^2] - 1 = 2\left(\frac{1}{6}n(n+1)(2n+1)\right) - 1$ ….(2)

By (1), $S_n = b_1 + b_2 + \ldots + b_n = \left(\frac{1}{6}n(n+1)(2n+1)\right)^2$ ….(3)

By (2) , The last term of $b_{n-1} = \frac{1}{6}(n-1)(n)(2n-1) - 1$.

Hence $b_n = S_n - S_{n-1} = \left(\frac{1}{6}n(n+1)(2n+1)\right)^2 - \left(\frac{1}{6}(n-1)(n)(2n-1)\right)^2 = \frac{1}{3}\left[2n^5 + n^3\right]$

Hence $S_n = b_1 + b_2 + \ldots + b_n = \frac{1}{3}\left[2\sum_{r=1}^{n} r^5 + \sum_{r=1}^{n} r^3\right] = \frac{1}{3}\left[2\sum_{r=1}^{n} r^5 + \frac{1}{4}n^2(n+1)^2\right]$ ….(4)

Compare (3) and (4), we get $\sum_{r=1}^{n} r^5 = \frac{1}{2}\left[3\left(\frac{1}{6}n(n+1)(2n+1)\right)^2 - \frac{1}{4}n^2(n+1)^2\right] = \frac{1}{12}n^2(n+1)^2(2n^2 + 2n - 1)$

7. (Backward Mathematical Induction)

(i) (a) I(n) : If $x_i \in [a, b]$, $i = 1, 2, \ldots, n$, then $f(x_1) + \ldots + f(x_n) \le nf\left(\frac{x_1 + \ldots + x_n}{n}\right)$

For $I(2^1)$, since it is given that $f(x_1) + f(x_2) \le 2f\left(\frac{x_1 + x_2}{2}\right)$. $\therefore$ $I(2^1)$ is true.

Assume $I(2^k)$ is true. i.e. $f(x_1)+...+f(x_{2^k}) \le 2^k f\left(\frac{x_1+...+x_{2^k}}{2^k}\right)$ ....(1)

For $I(2^{k+1})$, $f(x_1)+...+f(x_{2^k})+f(x_{2^k+1})+...+f(x_{2^{k+1}})$

$$\le 2^k f\left(\frac{x_1+...+x_{2^k}}{2^k}\right)+2^k f\left(\frac{x_{2^k+1}+...+x_{2^{k+1}}}{2^k}\right)=2^k\left[f\left(\frac{x_1+...+x_{2^k}}{2^k}\right)+f\left(\frac{x_{2^k+1}+...+x_{2^{k+1}}}{2^k}\right)\right], \text{ by (1)}$$

$$=2^k\left[f\left(\frac{x_1+...+x_{2^k}}{2^k}\right)+f\left(\frac{x_{2^k+1}+...+x_{2^{k+1}}}{2^k}\right)\right]\le 2^k 2\left[f\left(\frac{1}{2}\left(\frac{x_1+...+x_{2^k}}{2^k}+\frac{x_{2^k+1}+...+x_{2^{k+1}}}{2^k}\right)\right)\right], \text{ by I(2)}$$

$$=2^{k+1}\left[f\left(\frac{x_1+...+x_{2^k}+x_{2^k+1}+...+x_{2^{k+1}}}{2^{k+1}}\right)\right] \qquad \therefore I(2^{k+1}) \text{ is true.}$$

(b) Assume I(n) is true $(n \ge 2)$,

i.e. $f(x_1)+...+f(x_n) \le nf\left(\frac{x_1+...+x_n}{n}\right)=nf\left(\frac{n-1}{n}\left(\frac{x_1+...+x_{n-1}}{n-1}+\frac{x_n}{n-1}\right)\right)$

Put $x_n=\frac{x_1+...+x_{n-1}}{n-1}$, then

$$f(x_1)+...+f(x_{n-1})+f\left(\frac{x_1+...+x_{n-1}}{n-1}\right)\le nf\left(\frac{x_1+...+x_{n-1}}{n-1}\right)$$

$$f(x_1)+...+f(x_{n-1})\le (n-1)f\left(\frac{x_1+...+x_{n-1}}{n-1}\right) \qquad \therefore I(n-1) \text{ is also true.}$$

(c) $\forall n \in \mathbf{N}$, $\exists$ ($k \in \mathbf{N}$ and $r \in \mathbf{N}$) such that $n = 2^k - r$.

By (a), $I(2^k)$ is true $\Rightarrow$ $I(2^k-1)$ is true $\Rightarrow I(2^k-2)$ is true $\Rightarrow ... \Rightarrow$ $I(2^k-r)=I(n)$ is true.

(ii) $\sin x_1 + \sin x_2 = 2\sin\left(\frac{x_1+x_2}{2}\right)\cos\left(\frac{x_1-x_2}{2}\right)\le 2\sin\left(\frac{x_1+x_2}{2}\right)$, $\quad -\pi \le x_1 - x_2 \le \pi$.

$\therefore$ f(x) = sin x is convex on $[0, \pi]$. Last part follows from (i).

9. Let P(n) be the proposition : $\left(1-\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{6}-\frac{1}{8}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n}\right)=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{2n-1}-\frac{1}{2n}\right)$

For P(1), L.H.S. = $1-\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$, R.H.S. = $\frac{1}{2}\left(1-\frac{1}{2}\right)=\frac{1}{4}$, $\therefore$ P(1) is true.

Assume P(k) is true for some $k \in \mathbf{N}$, that is,

$$\left(1-\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{6}-\frac{1}{8}\right)+\cdots+\left(\frac{1}{2k-1}-\frac{1}{4k-2}-\frac{1}{4k}\right)=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{2k-1}-\frac{1}{2k}\right) \text{.........(1)}$$

For P(k + 1), $\left(1-\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{6}-\frac{1}{8}\right)+\cdots+\left(\frac{1}{2k-1}-\frac{1}{4k-2}-\frac{1}{4k}\right)+\left(\frac{1}{2k+1}-\frac{1}{4k+2}-\frac{1}{4k+4}\right)$

$$=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{2k-1}-\frac{1}{2k}\right)+\left(\frac{1}{2k+1}-\frac{1}{4k+2}-\frac{1}{4k+4}\right)$$

$$=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{2k-1}-\frac{1}{2k}\right)+\left[\left(\frac{1}{2k+1}-\frac{1}{2}\times\frac{1}{2k+1}\right)-\frac{1}{2}\times\frac{1}{2k+2}\right]$$

$$=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{2k-1}-\frac{1}{2k}\right)+\left[\frac{1}{2}\times\frac{1}{2k+1}-\frac{1}{2}\times\frac{1}{2k+2}\right]$$

$= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{2k-1} - \frac{1}{2k} + \frac{1}{2k+1} - \frac{1}{2k+2}\right)$, $\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true for all natural numbers n.

12. Let $F_n(z) = \frac{q}{1-q}(1-z) + \frac{q^2}{1-q^2}(1-z)(1-qz) + \ldots + \frac{q^n}{1-q^n}(1-z)(1-qz)\ldots(1-q^{n-1}z)$ **,**

Let P(n) be the proposition : $1 + F_n(z) - F_n(qz) = (1-qz)(1-q^2z)\ldots(1-q^nz)$.

For P(1), $1 + F_1(z) - F_1(qz) = 1 + \frac{q}{1-q}(1-z) - \frac{q}{1-q}(1-qz) = 1 - qz$ $\therefore$ P(1) is true.

Assume P(k) is true for some $k \in \mathbf{N}$, that is,

$1 + F_k(z) - F_k(qz) = (1-qz)(1-q^2z)\ldots(1-q^kz)$ ………(1)

For P(k + 1),

But, $F_{k+1}(z) = F_k(z) + \frac{q^{k+1}}{1-q^{k+1}}(1-z)(1-qz)\ldots(1-q^kz)$

and $F_{k+1}(qz) = F_k(qz) + \frac{q^{k+1}}{1-q^{k+1}}(1-qz)(1-q^2z)\ldots(1-q^{k+1}z)$

$1 + F_{k+1}(z) - F_{k+1}(qz)$

$= 1 + F_k(z) - F_k(qz) + \frac{q^{k+1}}{1-q^{k+1}}(1-z)(1-qz)\ldots(1-q^kz) - \frac{q^{k+1}}{1-q^{k+1}}(1-qz)(1-q^2z)\ldots(1-q^{k+1}z)$

$= (1-qz)(1-q^2z)\ldots(1-q^kz) + \frac{q^{k+1}}{1-q^{k+1}}(1-z)(1-qz)\ldots(1-q^kz) - \frac{q^{k+1}}{1-q^{k+1}}(1-qz)(1-q^2z)\ldots(1-q^{k+1}z)$, by (1)

$= (1-qz)(1-q^2z)\ldots(1-q^kz) + \frac{q^{k+1}}{1-q^{k+1}}(1-qz)\ldots(1-q^kz)[(1-q^kz) - (1-q^{k+1}z)]$

$= (1-qz)(1-q^2z)\ldots(1-q^kz) - q^{k+1}(1-qz)\ldots(1-q^kz)$

$= (1-qz)(1-q^2z)\ldots(1-q^kz)(1-q^{k+1}z)$

$\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall n \in \mathbf{N}$.

14. $1 + \frac{1}{a} = \frac{a+1}{a}$, $1 + \frac{1}{a} + \frac{a+1}{ab} = \frac{a+1}{a} + \frac{a+1}{ab} = \frac{(a+1)(b+1)}{ab}$

Assume that $1 + \frac{1}{a} + \frac{a+1}{ab} + \cdots + \frac{(a+1)(b+1)\ldots(s+1)}{abc\ldots sk} = \frac{(a+1)(b+1)\ldots(s+1)(k+1)}{abc\ldots sk}$

Adding the term $\frac{(a+1)(b+1)\ldots(s+1)(k+1)}{abc\ldots skl}$ to both sides, we get:

$1 + \frac{1}{a} + \frac{a+1}{ab} + \cdots + \frac{(a+1)(b+1)\ldots(s+1)}{abc\ldots sk} + \frac{(a+1)(b+1)\ldots(s+1)(k+1)}{abc\ldots skl} = \frac{(a+1)(b+1)\ldots(s+1)(k+1)}{abc\ldots sk} + \frac{(a+1)(b+1)\ldots(s+1)(k+1)}{abc\ldots skl}$

$= \frac{(a+1)(b+1)\ldots(s+1)(k+1)(l+1)}{abc\ldots skl}$

17. Let P(n) be the proposition : $\frac{n}{2n+1}+\left[\frac{1}{2^3-2}+\ldots+\frac{1}{(2n)^3-2n}\right]=\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}$

For P(1), L.H.S. = $\frac{1}{2\times1+1}+\frac{1}{2^3-2}=\frac{1}{2}=\frac{1}{1+1}$ = R.H.S. $\therefore$ P(1) is true.

Assume P(k) is true for some k$\in$ **N**, that is, $\frac{k}{2k+1}+\left[\frac{1}{2^3-2}+\ldots+\frac{1}{(2k)^3-2k}\right]=\frac{1}{k+1}+\frac{1}{k+2}+\ldots+\frac{1}{2k}$

or $\frac{k}{2k+1}+\left[\frac{1}{2^3-2}+\ldots+\frac{1}{(2k)^3-2k}\right]-\left(\frac{1}{k+1}+\frac{1}{k+2}+\ldots+\frac{1}{2k}\right)=0$ (*)

For P(k + 1), $\frac{k+1}{2k+3}+\left[\frac{1}{2^3-2}+\ldots+\frac{1}{[2(k+1)]^3-2(k+1)}\right]-\left(\frac{1}{k+2}+\frac{1}{k+3}+\ldots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}\right)$

$=\frac{k+1}{2k+3}-\frac{k}{2k+1}+\frac{1}{[2(k+1)]^3-2(k+1)}-\left(-\frac{1}{k+1}+\frac{1}{2k+1}+\frac{1}{2k+2}\right)$, by (*)

$=\frac{k+1}{2k+3}-\frac{k}{2k+1}+\frac{1}{2(k+1)(2k+1)(2k+3)}+\frac{1}{k+1}-\frac{1}{2k+1}-\frac{1}{2(k+1)}$

$=\frac{1}{(2k+1)(2k+3)}+\frac{1}{2(k+1)(2k+1)(2k+3)}-\frac{1}{2(k+1)(2k+1)}$

$=\frac{2(k+1)+1-(2k+3)}{2(k+1)(2k+1)(2k+3)}=0$ $\therefore$ P(k+1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall$ n $\in$ **N**.

18. Let P(n) be the proposition : $a_n=a+\frac{2}{3}(b-a)\left(1-\frac{1}{4^n}\right)$ (1) , $b_n=a+\frac{2}{3}(b-a)\left(1+\frac{1}{2\times4^n}\right)$ (2)

For P(1), $a_1=\frac{a+b}{2}=a+\frac{2}{3}(b-a)\left(1-\frac{1}{4}\right)$, $b_1=\frac{a_1+b}{2}=\frac{(a+b)/2+b}{2}=\frac{a+3b}{2}=a+\frac{2}{3}(b-a)\left(1+\frac{1}{2\times4}\right)$

$\therefore$ P(1) is true.

Assume P(k) is true for some k$\in$ **N**, i.e., $a_k=a+\frac{2}{3}(b-a)\left(1-\frac{1}{4^k}\right)$ (3) , $b_k=a+\frac{2}{3}(b-a)\left(1+\frac{1}{2\times4^k}\right)$ (4)

For P(k + 1),

$a_{k+1}=\frac{a_k+b_k}{2}=\frac{1}{2}\left[a+\frac{2}{3}(b-a)\left(1-\frac{1}{4^k}\right)+a+\frac{2}{3}(b-a)\left(1+\frac{1}{2\times4^k}\right)\right]$, by (3) and (4)

$=a+\frac{1}{3}(b-a)\left[\left(1-\frac{1}{4^k}\right)+\left(1+\frac{1}{2\times4^k}\right)\right]=a+\frac{2}{3}(b-a)\left(1-\frac{1}{4^{k+1}}\right)$ ….(5)

$b_{k+1}=\frac{a_{k+1}+b_k}{2}=\frac{1}{2}\left[a+\frac{2}{3}(b-a)\left(1-\frac{1}{4^{k+1}}\right)+a+\frac{2}{3}(b-a)\left(1+\frac{1}{2\times4^k}\right)\right]$,by (5) and (4)

$=a+\frac{1}{3}(b-a)\left[\left(1-\frac{1}{4^{k+1}}\right)+\left(1+\frac{2}{4^{k+1}}\right)\right]=a+\frac{2}{3}(b-a)\left(1+\frac{1}{2\times4^{k+1}}\right)$ $\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall$ n $\in$ **N**.

19. Let P(n) be the proposition : $p_{n-1} > p_n > k > q_n > q_{n-1} > 0$ where $p_0 > k > q_0$, $p_n = \frac{1}{2}(p_{n-1} + q_{n-1})$ and $p_n q_n = k^2$.

For P(1), Note that $p_0 > k > 0, q_0 = \frac{k^2}{p_0} > 0$.

$$p_1 = \frac{1}{2}(p_0 + q_0) < \frac{1}{2}(p_0 + p_0) = p_0, \quad q_1 = \frac{k^2}{p_1} > \frac{k^2}{p_0} = q_0$$

$$p_1 = \frac{1}{2}(p_0 + q_0) > \sqrt{p_0 q_0} = \sqrt{k^2} = k, \quad k > 0 \ , \qquad q_1 = \frac{k^2}{p_1} < \frac{k^2}{k} = k$$

Also, $p_1 > k > 0, q_1 = \frac{k^2}{p_1} > 0$

$\therefore p_0 > p_1 > k > q_1 > q_0 > 0$ and P(1) is true.

Assume P(m) is true for some m∈ **N**, i.e., $p_{m-1} > p_m > k > q_m > q_{m-1} > 0$ ... (1)

For P(m + 1), $p_{m+1} = \frac{1}{2}(p_m + q_m) < \frac{1}{2}(p_m + p_m) = p_m, \quad q_{m+1} = \frac{k^2}{p_{m+1}} > \frac{k^2}{p_m} = q_m$

$$p_{m+1} = \frac{1}{2}(p_m + q_m) > \sqrt{p_m q_m} = \sqrt{k^2} = k, \quad k > 0 \ , \qquad q_{m+1} = \frac{k^2}{p_{m+1}} < \frac{k^2}{k} = k$$

Also, $p_{m+1} > k > 0, q_{m+1} = \frac{k^2}{p_{m+1}} > 0$

∴ P(m + 1) is true.

By the Principle of Mathematical Induction, P(n) is true ∀ n ∈ **N**.

20. **(a)** $u_n(x) = \frac{x(x+1)(x+2)\dots(x+n-1)}{n!}$

Let S(p) be the statement : $\sum_{n=1}^{p} u_n(x) = u_p(x+1) - 1$

For S(1), $\sum_{n=1}^{1} u_n(x) = u_1(x) = \frac{x}{1!} = x, \qquad u_1(x+1) - 1 = \frac{x+1}{1!} - 1 = x$

∴ S(1) is true.

Assume S(k) is true for some k∈ **N**, i.e., $\sum_{n=1}^{k} u_n(x) = u_k(x+1) - 1$ ... (1)

For S(k + 1), $\sum_{n=1}^{k+1} u_n(x) = \sum_{n=1}^{k} u_n(x) + u_{k+1}(x) = u_k(x+1) - 1 + u_{k+1}(x)$, by (1)

$$= \frac{(x+1)(x+2)\dots(x+k)}{k!} - 1 + \frac{x(x+1)(x+2)\dots(x+k)}{(k+1)!}$$

$$= \frac{(x+1)(x+2)\dots(x+k)}{(k+1)!}[(k+1) + x] - 1$$

$$= \frac{(x+2)(x+3)\dots(x+k+1)(x+k+1)}{(k+1)!} - 1$$

$= u_{k+1}(x+1) - 1$, ∴ S(k + 1) is true.

By the Principle of Mathematical Induction, S(p) is true ∀ p ∈ **N**.

(b) Obviously, $u_n\left(\frac{1}{2}\right) = \frac{\left(\frac{1}{2}\right)\left(\left(\frac{1}{2}\right)+1\right)\left(\left(\frac{1}{2}\right)+2\right)\dots\left(\left(\frac{1}{2}\right)+n-1\right)}{n!} > 0$

Let P(n) be the proposition : $u_n\left(\frac{1}{2}\right) < \frac{1}{\sqrt{2n+1}}$

For P(1), $\quad$ $\text{L.H.S.} = u_1\left(\frac{1}{2}\right) = \frac{\left(\frac{1}{2}\right)}{1!} = \frac{1}{2}$, $\quad \text{R.H.S.} = \frac{1}{\sqrt{3}} \approx 0.5773502691896$, $\quad \therefore$ P(1) is true.

Assume P(k) is true for some k$\in$ **N**, i.e., $\quad u_k\left(\frac{1}{2}\right) < \frac{1}{\sqrt{2k+1}}$ ... (1)

For P(k + 1), $\quad u_{k+1}\left(\frac{1}{2}\right) = \frac{\left(\frac{1}{2}\right)\left(\left(\frac{1}{2}\right)+1\right)\left(\left(\frac{1}{2}\right)+2\right)\ldots\left(\left(\frac{1}{2}\right)+k-1\right)\left(\left(\frac{1}{2}\right)+k\right)}{(k+1)!} = \frac{\left(\frac{1}{2}\right)\left(\left(\frac{1}{2}\right)+1\right)\left(\left(\frac{1}{2}\right)+2\right)\ldots\left(\left(\frac{1}{2}\right)+k-1\right)}{k!} \times \frac{\left(\left(\frac{1}{2}\right)+k\right)}{k+1}$

$$< \frac{1}{\sqrt{2k+1}} \times \frac{\left(\left(\frac{1}{2}\right)+k\right)}{k+1} = \frac{1}{\sqrt{2k+1}} \times \frac{2k+1}{2(k+1)} = \frac{1}{\sqrt{2k+3}} \times \left[\frac{\sqrt{2k+1}\sqrt{2k+3}}{2k+2}\right] = \frac{1}{\sqrt{2k+3}} \times \left[\frac{\sqrt{4k^2+8k+3}}{2k+2}\right]$$

$$< \frac{1}{\sqrt{2k+3}} \times \left[\frac{\sqrt{4k^2+8k+4}}{2k+2}\right] = \frac{1}{\sqrt{2k+3}} \times \left[\frac{2k+2}{2k+2}\right] = \frac{1}{\sqrt{2k+3}}$$ $\quad \therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall$ n $\in$ **N**.

Since $\quad 0 < u_p\left(\frac{1}{2}\right) < \frac{1}{\sqrt{2p+1}}$, hence $\quad -1 < u_p\left(\frac{1}{2}\right) - 1 < \frac{1}{\sqrt{2p+1}} - 1$

Now, from **(a)**, put $x = -\frac{1}{2}$, $\quad \sum_{n=1}^{p} u_n\left(-\frac{1}{2}\right) = u_p\left(1-\frac{1}{2}\right) - 1 = u_p\left(\frac{1}{2}\right) - 1$

$$-1 < \sum_{n=1}^{p} u_n\left(-\frac{1}{2}\right) < \frac{1}{\sqrt{2p+1}} - 1$$

$$-1 < \frac{\left(-\frac{1}{2}\right)}{1!} + \frac{\left(-\frac{1}{2}\right)\left(\left(-\frac{1}{2}\right)+1\right)}{2!} + \cdots + \frac{\left(-\frac{1}{2}\right)\left(\left(-\frac{1}{2}\right)+1\right)\left(\left(-\frac{1}{2}\right)+2\right)\ldots\left(\left(-\frac{1}{2}\right)+p-1\right)}{p!} < \frac{1}{\sqrt{2p+1}} - 1$$

$$-1 < -\left[\frac{1}{2} + \frac{1}{2\cdot4} + \frac{1\cdot3}{2\cdot4\cdot6} + \cdots + \frac{1\cdot3\cdot5\cdots(2p-3)}{2\cdot4\cdot6\cdots(2p)}\right] < \frac{1}{\sqrt{2p+1}} - 1$$

$$1 > \frac{1}{2} + \frac{1}{2\cdot4} + \frac{1\cdot3}{2\cdot4\cdot6} + \cdots + \frac{1\cdot3\cdot5\cdots(2p-3)}{2\cdot4\cdot6\cdots(2p)} > 1 - \frac{1}{\sqrt{2p+1}}$$

21. Let P(n) be the proposition : $\quad \ell_n < \sqrt{a} + 1$ and $\ell_{n-1} < \ell_n$.

For P(1), $\quad \ell_1 = \sqrt{a} < \sqrt{a} + 1$ and $\ell_1 = \sqrt{a} < \sqrt{a + \sqrt{a}} = \ell_2$ $\quad \therefore$ P(1) is true.

Assume P(k) is true for some k$\in$ **N**, i.e., $\quad \ell_k < \sqrt{a} + 1$ (1) $\quad$ and $\quad \ell_{k-1} < \ell_k$ (2)

For P(k + 1), $\quad \ell_{k+1} = \sqrt{a + \ell_k} < \sqrt{a + \sqrt{a} + 1} < \sqrt{a + 2\sqrt{a} + 1} = \sqrt{a} + 1$

and $\quad \ell_{k+2} = \sqrt{a + \ell_{k+1}} > \sqrt{a + \ell_k} = \ell_{k+1}$

$\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall$ n $\in$ **N**.

22. **(a)** Prove of $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$ is omitted.

$1\cdot n + 2\cdot(n-1) + 3\cdot(n-2) + \cdots + n\cdot 1 = \sum_{i=1}^{n} i(n+1-i) = (n+1)\sum_{i=1}^{n} i - \sum_{i=1}^{n} i^2$

$= (n+1)\frac{n(n+1)}{2} - \frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}n(n+1)(n+2)$ ….(1)

(b) Let P(n) be the proposition : $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} \le 2\sqrt{n}$ ….(2)

For P(1), L.H.S. = 1 ≤ 2 = R.H.S. ∴ P(1) is true.

Assume P(k) is true for some $k \in \mathbf{N}$, i.e., $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{k}} \le 2\sqrt{k}$ ….(*)

For P(k + 1), $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \le 2\sqrt{k} + \frac{1}{\sqrt{k+1}}$, by (*)

$= 2\sqrt{k+1} - 2\sqrt{k+1} + 2\sqrt{k} + \frac{1}{\sqrt{k+1}} = 2\sqrt{k+1} - \frac{2(k+1) - 2\sqrt{k(k+1)} - 1}{\sqrt{k+1}}$

$= 2\sqrt{k+1} - \frac{(k+1) - 2\sqrt{k(k+1)} + k}{\sqrt{k+1}} = 2\sqrt{k+1} - \frac{\left(\sqrt{k+1} - \sqrt{k}\right)^2}{\sqrt{k+1}} \ge 2\sqrt{k+1}$ ∴ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall\, n \in \mathbf{N}$.

By (1),

$\frac{1}{6}(n+1)(n+2) = \frac{1.n + 2(n-1) + 3(n-2) + ... + n.1}{n} > [(1.n)(2.(n-1))...(n.1)]^{1/n} = \left[(n!)^2\right]^{1/n}$, by A.M.>G.M.

$\therefore\ n! < \left[\frac{1}{6}n(n+1)(2n+1)\right]^{\frac{n}{2}}$

By (2), $\frac{2\sqrt{n}}{n} \ge \frac{1}{n}\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}}\right) > \left(1 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{3}} \times ... \times \frac{1}{\sqrt{n}}\right)^{1/n} = \left(\frac{1}{n!}\right)^{1/2n}$

$\therefore\ \left(\frac{n}{4}\right)^n < n!$

23. Let P(n) be the proposition : $\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \cdots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$ where a_i is an arithmetic sequence.

For P(2), L.H.S.$= \frac{1}{\sqrt{a_1}+\sqrt{a_2}} =$ R.H.S., ∴ P(2) is true.

Assume P(k) is true for some $k \in \mathbf{N}\backslash\{1\}$, that is, $\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \cdots + \frac{1}{\sqrt{a_{k-1}}+\sqrt{a_k}} = \frac{k-1}{\sqrt{a_1}+\sqrt{a_k}}$………(1)

For P(k + 1), $\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \cdots + \frac{1}{\sqrt{a_{k-1}}+\sqrt{a_k}} + \frac{1}{\sqrt{a_k}+\sqrt{a_{k+1}}} = \frac{k-1}{\sqrt{a_1}+\sqrt{a_k}} + \frac{1}{\sqrt{a_k}+\sqrt{a_{k+1}}}$, by (1)

$= \frac{(k-1)(\sqrt{a_1}-\sqrt{a_k})}{(\sqrt{a_1}+\sqrt{a_k})(\sqrt{a_1}-\sqrt{a_k})} + \frac{(\sqrt{a_k}-\sqrt{a_{k+1}})}{(\sqrt{a_k}+\sqrt{a_{k+1}})(\sqrt{a_k}-\sqrt{a_{k+1}})} = \frac{(k-1)(\sqrt{a_1}-\sqrt{a_k})}{a_1-a_k} + \frac{(\sqrt{a_k}-\sqrt{a_{k+1}})}{a_k-a_{k+1}}$

$= \frac{(k-1)(\sqrt{a_1}-\sqrt{a_k})}{-(k-1)d} + \frac{(\sqrt{a_k}-\sqrt{a_{k+1}})}{-d}$, where d is the common difference.

$$= -\frac{1}{d}(\sqrt{a_1} - \sqrt{a_k}) - \frac{1}{d}\left(\sqrt{a_k} - \sqrt{a_{k+1}}\right) = -\frac{1}{d}\left(\sqrt{a_1} - \sqrt{a_{k+1}}\right)$$

$$= \frac{k}{-kd}\left(\sqrt{a_1} - \sqrt{a_{k+1}}\right) = \frac{k}{a_1 - a_{k+1}}\left(\sqrt{a_1} - \sqrt{a_{k+1}}\right) = \frac{k}{\left(\sqrt{a_1}+\sqrt{a_{k+1}}\right)\left(\sqrt{a_1}-\sqrt{a_{k+1}}\right)}\left(\sqrt{a_1} - \sqrt{a_{k+1}}\right)$$

$$= \frac{k}{\sqrt{a_1}+\sqrt{a_{k+1}}}$$

$\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall$ n $\in$ **N\{1}.**